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Estimates of the norms of the Toeplitz operators of H^∞ determined by rational inner functions

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Abstract

Let D denote the unit disc and T the unit circle. Let $E \subseteq D$ and $b_n(E)$ be the totality of all finite Blaschke products B_n with n zeros belonging to E . Let

$$\Omega_n(E) = \sup \left\{ \left\| \frac{1}{2\pi i} \int_T \frac{h(\varsigma)}{B_n(\varsigma)(\varsigma - z)} d\varsigma \right\|_{H^\infty} : \|h\|_{H^\infty} \leq 1, B_n \in b_n(E) \right\}$$

and $E_\xi = \{z : \varepsilon \leq |z| < 1, |z| = \xi\}$, $\xi \in T$, $\varepsilon > 0$. An elementary proof is given of the equalities

$$\Omega_n(D) = \Omega_n(E_\xi) = 1 + 2n,$$

for all $\xi \in T$, $\varepsilon > 0$.

1 Introduction

Let D denote the unit disc and T the unit circle. Let H^∞ denote the space of all functions analytic in D and such that $\|f\|_{H^\infty} = \sup_{z \in D} |f(z)| < \infty$. For $f \in L^\infty(T)$, we denote by T_f the Toeplitz operator on H^∞ , defined by

$$T_f h = \int_T \frac{\overline{f(\varsigma)} h(\varsigma)}{\varsigma - z} \varsigma dm(\varsigma), \quad h \in H^\infty.$$

Here m denotes normalised Lebesgue measure on T .

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Let $E \subseteq D$ and $b_n(E)$ be the totality of all finite Blaschke products B_n with n zeros belonging to E . Let

$$\Omega_n(E) = \sup \{ \|T_{B_n}\|_{H^\infty} : B_n \in b_n(E) \}.$$

In the present paper we shall give an elementary proof of the equality

$$\Omega_n(D) = 1 + 2n,$$

with more complicated method was proven by Pekarski [1].

Previously in [2] we proved that $1 + 2n \leq \Omega_n(D) \leq 1 + \pi n$.

2 Main results

Our main result is based on the following lemmas. Let $f \in H^\infty$ and

$$\Lambda(f) = \sup_{\eta \in T} \int_T \frac{|f(\varsigma\eta) - f(\bar{\varsigma}\eta)|}{|1 - \varsigma|} dm(\varsigma) < \infty.$$

Lemma 1. *If $f \in H^\infty$, then $\|T_f\|_{H^\infty} \leq \|f\|_{H^\infty} + \Lambda(f)$.*

Proof.

$$\begin{aligned} \|T_f\|_{H^\infty} &= \sup \left\{ \lim_{r \rightarrow 1-0} \left| \int_T \frac{\bar{f}(\varsigma)h(\varsigma)}{1 - \bar{\varsigma}r\eta} dm(\varsigma) \right| : \eta \in T, \|h\|_{H^\infty} \leq 1 \right\} = \\ &= \sup \left\{ \lim_{r \rightarrow 1-0} \left| \int_T \frac{\bar{f}(\varsigma\eta)h(\varsigma\eta)}{1 - r\bar{\varsigma}} dm(\varsigma) \right| : \eta \in T, \|h\|_{H^\infty} \leq 1 \right\} \leq \\ &\leq \sup \left\{ \lim_{r \rightarrow 1-0} \left| \int_T \frac{\bar{f}(\varsigma\eta) - \bar{f}(\bar{\varsigma}\eta)}{1 - r\bar{\varsigma}} h(\varsigma\eta) dm(\varsigma) \right| : \eta \in T, \|h\|_{H^\infty} \leq 1 \right\} + \|f\|_{H^\infty} \leq \\ &\leq \sup \left\{ \left| \int_T \frac{|f(\varsigma\eta) - f(\bar{\varsigma}\eta)|}{|1 - \varsigma|} dm(\varsigma) \right| : \eta \in T \right\} + \|f\|_{H^\infty} = \Lambda(f) + \|f\|_{H^\infty}. \end{aligned}$$

We used, that $g(z) = \bar{f}(\bar{z}\eta) \in H^\infty$ and

$$\left| \int_T \frac{\bar{f}(\bar{\varsigma}\eta)h(\varsigma\eta)}{1 - r\bar{\varsigma}} dm(\varsigma) \right| \leq \|f\|_{H^\infty} \|h\|_{H^\infty}.$$

□

Lemma 2. *If*

$$I(z) = \frac{z - a}{1 - z\bar{a}}, \quad a \in D, \quad \text{then } \Lambda(I) \leq 2.$$

Proof.

$$\begin{aligned} \Lambda(I) &= \sup_{\eta \in T} \int_T \left| \frac{\varsigma\eta - a}{1 - \varsigma\eta\bar{a}} - \frac{\bar{\varsigma}\eta - a}{1 - \bar{\varsigma}\eta\bar{a}} \right| \frac{dm(\varsigma)}{|1 - \varsigma|} = \\ &= \sup_{\eta \in T} \int_T \frac{(1 - |a|^2) |\varsigma - \bar{\varsigma}|}{|1 - \varsigma\eta\bar{a}| |1 - \bar{\varsigma}\eta\bar{a}| |1 - \varsigma|} dm(\varsigma) \leq 2 \sup_{\eta \in T} \int_T \frac{1 - |a|^2}{|1 - \varsigma\eta\bar{a}| |1 - \bar{\varsigma}\eta\bar{a}|} dm(\varsigma) \leq \\ &\leq 2 \sup_{\eta \in T} \left(\int_T \frac{1 - |a|^2}{|1 - \varsigma\eta\bar{a}|^2} dm(\varsigma) \right)^{1/2} \left(\int_T \frac{1 - |a|^2}{|1 - \bar{\varsigma}\eta\bar{a}|^2} dm(\varsigma) \right)^{1/2} = 2. \end{aligned}$$

□

Lemma 3. *If $I_k(z)$, $k = 1, 2, \dots, n$ is inner functions ($|I_k(\varsigma)| = 1$ a.e on T), then $\Lambda(I_1 I_2 \dots I_n) \leq \Lambda(I_1) + \Lambda(I_2) + \dots \Lambda(I_n)$.*

Proof. The proof follows at once from the identity

$$\begin{aligned} &I_1(\varsigma\eta)I_2(\varsigma\eta)\dots I_n(\varsigma\eta) - I_1(\bar{\varsigma}\eta)I_2(\bar{\varsigma}\eta)\dots I_n(\bar{\varsigma}\eta) = \\ &= I_1(\varsigma\eta)I_2(\varsigma\eta)\dots I_n(\varsigma\eta) - I_1(\bar{\varsigma}\eta)I_2(\varsigma\eta)\dots I_n(\varsigma\eta) + \\ &+ I_1(\bar{\varsigma}\eta)I_2(\varsigma\eta)\dots I_n(\varsigma\eta) - I_1(\bar{\varsigma}\eta)I_2(\bar{\varsigma}\eta)\dots I_n(\varsigma\eta) + \\ &\dots\dots\dots \\ &+ I_1(\bar{\varsigma}\eta)I_2(\bar{\varsigma}\eta)\dots I_{n-1}(\bar{\varsigma}\eta)I_n(\varsigma\eta) - I_1(\bar{\varsigma}\eta)I_2(\bar{\varsigma}\eta)\dots I_{n-1}(\bar{\varsigma}\eta)I_n(\bar{\varsigma}\eta) = \\ &= (I_1(\varsigma\eta) - I_1(\bar{\varsigma}\eta)) I_2(\varsigma\eta)\dots I_n(\varsigma\eta) + \\ &\quad (I_2(\varsigma\eta) - I_2(\bar{\varsigma}\eta)) I_1(\bar{\varsigma}\eta)\dots I_n(\varsigma\eta) + \\ &\quad \dots\dots\dots \\ &+ (I_n(\varsigma\eta) - I_n(\bar{\varsigma}\eta)) I_1(\bar{\varsigma}\eta)I_2(\bar{\varsigma}\eta)\dots I_{n-1}(\bar{\varsigma}\eta). \end{aligned}$$

□

Lemma 4. *If $B_n \in b_n(E)$, then $\Lambda(B_n) \leq 2$.*

Proof. Let

$$B_n(z) = \prod_{k=1}^n \frac{z - a_k}{1 - z\overline{a_k}}, \quad a_k \in D.$$

From Lemmas 2 and 3 it follows that

$$\Lambda(B_n) \leq \sum_{k=1}^n \Lambda\left(\frac{z - a_k}{1 - z\overline{a_k}}\right) \leq 2n.$$

□

Theorem 1. $\Omega_n(D) = \Omega_n(E_\xi) = 1 + 2n$ for all $\xi \in T$,

where $E_\xi = \{z : \varepsilon \leq |z| < 1, |z|/|z| = \xi\}$, $\varepsilon > 0$.

Proof. From Lemmas 1 and 4 follows that

$$\begin{aligned} \Omega_n(E_\xi) &\leq \Omega_n(D) = \sup \{\|T_{B_n}\|_{H^\infty} : B_n \in b_n(D)\} \leq \\ &\leq 1 + \sup \{\Lambda(B_n) : B_n \in b_n(D)\} \leq 1 + 2n. \end{aligned}$$

Let $\xi \in T$. We will show that

$$\Omega_n(D) = \Omega_n(E_\xi) = 1 + 2n.$$

Let

$$x_k = (1 - q^k)\xi, \quad 0 < q < 1, \quad \varepsilon < 1 - q, \quad B_n(z) = \prod_{k=1}^n \frac{z - x_k}{1 - z\overline{x_k}}.$$

Let $m > n$ and

$$y_k = B'_n(x_k)(|x_k|^2 - 1)\xi, \quad k \neq m, \quad y_m = B_n(x_m).$$

Since

$$|B'_n(z)|(1 - |z|^2) \leq 1, \quad |B_n(z)| \leq 1, \quad (z \in D),$$

then $|y_k| \leq 1$ and by a well known Carleson interpolation theorem there exists a function $h_0 \in H^\infty$, such that

$$h_0(x_k) = y_k = B'_n(x_k)(|x_k|^2 - 1)\xi, \quad k \neq m,$$

$$h_0(x_m) = y_m = B_n(x_m),$$

$$\|h_0\|_{H^\infty} \leq A(q),$$

where

$$A(q) \rightarrow 1 \quad \text{as } q \rightarrow 0 \quad [3].$$

Since $B_n \in b_n(E_\xi)$, then

$$\begin{aligned}\Omega_n(E_\xi) &\geq \|T_{B_n}\|_{H^\infty} \geq \frac{1}{A(q)} \frac{1}{2\pi} \left| \int_T \frac{h_0(\varsigma)}{B_n(\varsigma)(\varsigma - x_m)} d\varsigma \right| = \\ &= \frac{1}{A(q)} \left| \frac{h_0(x_m)}{B_n(x_m)} + \sum_{k=1}^n \frac{h_0(x_k)}{B'_n(x_k)(x_k - x_m)} \right| = \frac{1}{A(q)} \left| 1 + \sum_{k=1}^n \frac{|x_k|^2 - 1}{|x_k| - |x_m|} \right|.\end{aligned}$$

Since $m > n$ can be every arbitrary long positive integer and $|x_m| \rightarrow 1$ as $m \rightarrow \infty$, then

$$\Omega_n(E_\xi) \geq \frac{1}{A(q)} \sum_{k=1}^n (1 + |x_k|) = \frac{1}{A(q)} \left(1 + 2n - \sum_{k=1}^n q^k \right).$$

Using the fact that $A(q) \rightarrow 1$ as $q \rightarrow 0$, we obtain $\Omega_n(E_\xi) \geq 1 + 2n$. This implies $\Omega_n(D) = \Omega_n(E_\xi) = 1 + 2n$ for all $\xi \in T$.

□

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